**Path Integral Formulation of Propagator**

Let’s work these out for bosons/fermions separately. First we have to discuss coherent states – i.e. states which diagonalize the creation/annihilation operators.

**Coherent States of bosons and fermions**

First we gotta discuss coherent states. We’ll start with bosons. The basic work has already been done in the QM Time-Dependent HO (single particle) file. We’ll translate the work on the HO to the present context. Consider an H with a single ‘state’, populated/depopulated by creation/annihilation operators ψ†, ψ. And we want the eigenstates/values of these operators.



We know from our work that the eigenvalues, ψ (ψ\*), are any complex number. And the state corresponding to the specified complex number is:



where |0> is the 0 occupation number state. And the overlap between two states is:



and finally, the resolution of identity is given by:



And the matrix elements of normal ordered operator A(a†,a):



It stands to reason, that we can generalize to many body states quite naturally. Let |ψ> denote a ‘field’ with some ψ value, ψ(x) at every point, x. If a boson field then ψ(x) will comprise some specified set of complex numbers, and if fermion field, then of Grassman numbers. So that:



(and for Grassman numbers, it’s more common to denote the complex conjugate as ) Just to be clear – any set of numbers, distributed over all space (i.e., any field) you can dream up – will work. Then we have the generalized translation operator turns into a functional operator (basically a separate operator for each point in space – all multiplied together, which correlates to a sum/integral in the exponent) …



So given our specified set of eigenvalues, ψ(x), we can construct the corresponding field |ψ>. And the states have an overlap:



And according to our work in the QM/Time-Dependent/HO file, we’ll surmise that a complete set of states is:



And the matrix elements of a normal ordered functional operator A(ψ†,ψ):



Not much changes, formally, in the fermionic case. We can write down a many-body eigenstate |ψ> of the annihilation operator, , so that.



Obviously, as before



But we’ll observe that the eigenvalues, ψ(x), of the state |ψ> must be Grassman numbers. Because for instance,



and complex numbers do not anti-commute. This is the defining property of Grassman numbers…I guess. It follows that ψ(x)2 = 0.

**Propagator**

First let’s consider the many-body propagator:



where ψb and ψa are many-body coherent states. It doesn’t look like it will be necessary to distinguish between bosons and fermions in what follows. We’ll just default to Fermions I guess, but if bosons then ignore the spin sum part. Say the Hamiltonian is:



Normally we insert resolutions of identity that diagonalize the operators – the position/momentum operator. This time we’ll fill in the bosonic/fermionic coherent state resolution of identity. Since we have a spin up and spin down operator, our identity resolution will be:



where we use the last expression as shorthand for the former,



This will give us, generalizing our result above to allow for multiple spin indices:



And now we’ll group the overlap terms together, and the H’s together.



Now to first order in δt, it seems we can change some of the ψ’s. For instance, change ψσ,b\* → ψσ,n-1\* in the top line, ψσ,n-1\* → ψσ,n-2\* in the second line, etc. And in the last line, I’m going to do the same thing. Since the error is first order in δt, that makes the overall error in the exponent (δt)2, and therefore negligible in the small δt limit.



and now we can introduce derivatives,



Taking the continuum limit, we can write this as a functional integral, and we have:



where the integration bounds are to indicate that we have the boundary conditions ψσ(x,0) = ψσ,a(x) and ψσ(x,t) = ψσ,b(x). And



The combination of derivatives is just Im[ψ\*∂tψ]. But an integration by parts on say the first term, will make this:



and just a reminder, if these are fermions indeed, then proper notation would be ψ\*σ → σ, and if we have rather bosons, then shouldn’t have a spin sum. Maybe should go back and make sure that if we did any commuting of numbers in the derivation above, that it still works if we incur the (-) cost of commuting fermion/Grassman numbers.